PPLATO PROMOTING PHYSICS LEARNING & TEACHING OPPORTUNITIES

Basic Mathematics

Fractions

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The aim of this document is to provide a short, self-assessment programme for students who wish to acquire a basic competence in the use of fractions.

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1. Fractions (Introduction)

The first thing to note is that all fractions can be represented in many different ways. Thus 3/12, 4/16, 5/20 represent the same fraction which, in its *lowest terms*, is 1/4.

Example 1 To show that each of the above fractions is equal to 1/4 we proceed as follows:

$$\frac{3}{12} = \frac{1 \times 3}{4 \times 3} = \frac{1 \times \cancel{3}}{4 \times \cancel{3}} = \frac{1}{4},$$

$$\frac{4}{16} = \frac{1 \times 4}{4 \times 4} = \frac{1 \times \cancel{4}}{4 \times \cancel{4}} = \frac{1}{4},$$

$$\frac{5}{20} = \frac{1 \times 5}{4 \times 5} = \frac{1 \times \cancel{5}}{4 \times \cancel{5}} = \frac{1}{4}.$$

The fractions in the left hand column each have a factor common to the numerator (top) and denominator (bottom) of that fraction, which is cancelled to give the fraction in its *lowest terms*.

In many problems different fractions need to be written to a common denominator and the *least common denominator* is often the best.

Example 2 In each case below, write each of the sets of fractions with their *least common denominator*.

$$\text{(a)} \quad \frac{2}{5}, \frac{3}{4}, \qquad \text{(b)} \quad \frac{1}{6}, \frac{2}{15}.$$

Solution

(a) In this case the denominators are 4 and 5, which have no factors in common. The smallest number which is a multiple of both is therefore $4\times5=20$. Then

$$\frac{2}{5} = \frac{2\times4}{5\times4} = \frac{8}{20} \qquad \text{and} \qquad \frac{3}{4} = \frac{3\times5}{4\times5} = \frac{15}{20}.$$

(b) In this case, $6=2\times3$ and $15=5\times3$. The least common denominator must have 2×3 and 5×3 as factors. The smallest such number is $2\times3\times5=30$. Then

$$\frac{1}{6} = \frac{1\times5}{6\times5} = \frac{5}{30} \qquad \text{and} \qquad \frac{2}{15} = \frac{2\times2}{15\times2} = \frac{4}{30}.$$

An example of a problem which is simplified by writing sets of fractions with a common denominator is

Example 3 Arrange each of the following fractions in increasing order of size.

$$\text{(a) } 3/4, \qquad \text{(b) } 2/3, \qquad \text{(c) } 5/6.$$

Solution The denominators this time are

$$4 = 2 \times 2, \qquad 3 \quad \text{and} \quad 6 = 2 \times 3.$$

The *least common denominator* of these is $2 \times 2 \times 3 = 12$.

$$\text{(a)} \qquad \frac{3}{4} \quad = \quad \frac{3 \times 3}{4 \times 3} \quad = \quad \frac{9}{12},$$

$$\text{(b)} \qquad \frac{2}{3} \quad = \quad \frac{2 \times 4}{3 \times 4} \quad = \quad \frac{8}{12},$$

$$\text{(c)} \qquad \frac{5}{6} \quad = \quad \frac{5 \times 2}{6 \times 2} \quad = \quad \frac{10}{12}.$$

The *order of size*, starting with the smallest, is 2/3, 3/4, 5/6.

EXERCISE 1. In each case below, arrange the fractions in increasing order of size. (Click on the green letters for solution.)

(a) $7/12$, $1/2$, $2/3$.   (b) $1/3$, $5/8$, $3/4$.

(c) $5/6$, $8/9$, $11/12$.   (d) $2/3$, $3/5$, $7/10$.

Now try this short quiz. (Work out your solution and click on the green letter to see if you are correct.)

Quiz Of the following sets of fractions, only one is in ascending order of size. Which one?

(a) $1/3$, $5/8$, $3/5$.   (b) $3/8$, $2/3$, $3/4$.

(c) $2/3$, $3/4$, $1/2$.   (d) $3/5$, $5/7$, $1/2$.

2. Addition of Fractions

In this section we look at the addition of fractions. If fractions are to be added then they must have the same denominators.

Example 4 Write the following sums of fractions as single fractions.

$$\text{(a)} \quad \frac{1}{8} + \frac{2}{3}, \qquad \text{(b)} \quad \frac{2}{3} - \frac{3}{4}.$$

Solution

(a) The denominators are 8 and 3, which have no common factors, so the *least common denominator* is 3×8=24.

$$\begin{aligned}
\frac{1}{8} + \frac{2}{3} &= \frac{1 \times 3}{8 \times 3} + \frac{2 \times 8}{3 \times 8} \\
&= \frac{3}{24} + \frac{16}{24} \\
&= \frac{3 + 16}{24} = \frac{19}{24}.
\end{aligned}$$

(b) As in **(a)**, the denominators 3 and 4 have no common factors so the *least common denominator* is 3×4=12.

$$\frac{2}{3} - \frac{3}{4} = \frac{2 \times 4}{3 \times 4} - \frac{3 \times 3}{4 \times 3}$$

$$= \frac{8}{12} - \frac{9}{12}$$

$$= \frac{8 - 9}{12} = -\frac{1}{12}.$$

The exercise below gives you some practise at addition and subtraction of fractions.

EXERCISE 2. Evaluate the following, putting the final answer in its lowest terms. (Click on the green letters for solution.)

(a) $\dfrac{1}{3} + \dfrac{1}{8}$,   (b) $\dfrac{5}{6} + \dfrac{2}{4}$,   (c) $\dfrac{3}{4} + \dfrac{4}{5}$,

(d) $\dfrac{4}{3} - \dfrac{1}{4}$,   (e) $\dfrac{5}{6} - \dfrac{3}{2}$,   (f) $\dfrac{5}{6} - \dfrac{3}{10}$.

To finish this section here are two quizzes.

Quiz Which of the following fractions is the result of evaluating the sum

$$\frac{3}{4} + \frac{1}{6} ?$$

(a) 11/12,   (b) 7/12,   (c) 5/12,   (d) 1/12.

Quiz From the fraction given below, choose the one which is exactly mid-way between 2/3 and 4/5. (Hint: first change the two fractions so that they have the same denominator.)

(a) 7/15,   (b) 3/5,   (c) 10/15   (d) 11/15.

3. Multiplication and Division of Fractions

To **multiply** two or more fractions together, multiply the numerators together and multiply the denominators together.

Example 5 Evaluate the following products.

$$\text{(a) } \frac{2}{5} \times \frac{3}{4}, \qquad \text{(b) } \frac{3}{4} \times \frac{1}{3}.$$

Solution

In each case the result is obtained by multiplying together the **numerators** and the **denominators**, as follows.

$$\text{(a) } \frac{2}{5} \times \frac{3}{4} = \frac{2 \times 3}{5 \times 4} = \frac{6}{20} = \frac{3}{10}, \qquad \text{(b) } \frac{3}{4} \times \frac{1}{3} = \frac{3 \times 1}{4 \times 3} = \frac{3}{12} = \frac{1}{4}.$$

The **common factors** have been cancelled each time to leave the fraction in its **lowest form**. **Always** try to do this.

> **Division** by a fraction of the form a/b
> is equivalent to
> **multiplication** by the fraction b/a.

Example 6 Write the fraction $1 \div \left(\frac{3}{4}\right)$ as a simple fraction.

Solution First note that

$$\left(\frac{3}{4}\right) \times \left(\frac{4}{3}\right) = \frac{3 \times 4}{4 \times 3} = \frac{12}{12} = 1 \,.$$

Divide both sides of this equation by $(3/4)$.

$$\left(\frac{3}{4}\right) \times \left(\frac{4}{3}\right) = 1 \,,$$

$$\therefore \left(\frac{4}{3}\right) = 1/\left(\frac{3}{4}\right) = 1 \div \left(\frac{3}{4}\right) \,,$$

so that $\quad 1 \div \left(\frac{3}{4}\right) = \left(\frac{4}{3}\right) \,.$

Example 7 Write the fraction $\left(\frac{1}{8}\right) \div \left(\frac{3}{4}\right)$ as a single fraction.

Solution In the previous example it was shown that $1 \div \left(\frac{3}{4}\right) = \left(\frac{4}{3}\right)$, and this result can be used here.

$$
\begin{aligned}
\left(\frac{1}{8}\right) \div \left(\frac{3}{4}\right) &= \frac{\left(\frac{1}{8}\right)}{\left(\frac{3}{4}\right)}, \\
&= \frac{\left(\frac{1}{8}\right) \times 1}{1 \times \left(\frac{3}{4}\right)}, \\
&= \frac{\left(\frac{1}{8}\right)}{1} \times \frac{1}{\left(\frac{3}{4}\right)}, \\
&= \left(\frac{1}{8}\right) \times \left(\frac{4}{3}\right), \quad (\text{using } \textbf{example 6}), \\
&= \frac{1}{2} \times \frac{1}{3} = \frac{1}{6}.
\end{aligned}
$$

where a common factor 4 has been cancelled.

Here is an exercise to provide you with some practice at using these techniques.

EXERCISE 3.
Evaluate each of the following fractions in their *lowest terms*.

(a) $\quad \dfrac{1}{5} \times \dfrac{10}{11}$,

(b) $\quad \dfrac{1}{5} \div \dfrac{2}{15}$,

(c) $\quad \left(\dfrac{1}{3} + \dfrac{1}{8}\right) \times \dfrac{2}{11}$,

(d) $\quad \left(\dfrac{5}{6} + \dfrac{1}{2}\right) \div \dfrac{1}{3}$,

(e) $\quad \dfrac{\left(\frac{3}{4} + \frac{4}{5}\right)}{\left(\frac{1}{4} - \frac{1}{5}\right)}$,

(f) $\quad \dfrac{\left(\frac{5}{6} - \frac{3}{2}\right)}{\left(\frac{4}{3} - \frac{1}{4}\right)}$.

The final part of this package is a quiz on fractions and this may be found on the next page.

4. Quiz on Fractions

Begin Quiz Evaluate each of the following and choose the answer from those given below.

1. $$\left(\tfrac{3}{4} - \tfrac{2}{3}\right) \div \left(\tfrac{3}{4}\right)$$

 (a) $2/9$,   (b) $1/6$, $5/12$, (c)   (d) $1/9$.

2. $$\left(\tfrac{3}{5} + \tfrac{1}{3}\right) \times \left(\tfrac{5}{7}\right)$$

 (a) $1/7$,   (b) $2/3$,   (c) $2/15$,   (d) $4/21$.

3. $$\left(\tfrac{3}{8} - \tfrac{1}{5}\right) \div \left(\tfrac{7}{10} - \tfrac{2}{3}\right)$$

 (a) $4/15$,   (b) $13/25$,   (c) $13/20$,   (d) $21/4$.

End Quiz

Solutions to Exercises

Exercise 1(a) The *least common denominator* of the fractions is 12, so

$$\frac{1}{2} \;=\; \frac{1 \times 6}{2 \times 6} \;=\; \frac{6}{12},$$

$$\frac{2}{3} \;=\; \frac{2 \times 4}{3 \times 4} \;=\; \frac{8}{12}.$$

Since 7/12 already has 12 as denominator, the required order of the fractions is

$$\frac{6}{12}, \; \frac{7}{12}, \; \frac{8}{12},$$

or

$$\frac{1}{2}, \; \frac{7}{12}, \; \frac{2}{3}.$$

Click on the green square to return

□

Exercise 1(b) In this case, the *least common denominator* of the fractions is 24.

$$\frac{1}{3} = \frac{1 \times 8}{3 \times 8} = \frac{8}{24},$$

$$\frac{5}{8} = \frac{5 \times 3}{8 \times 3} = \frac{15}{24},$$

$$\frac{3}{4} = \frac{3 \times 6}{4 \times 6} = \frac{18}{24}.$$

The fractions are thus increasing in the order in which they appear.

Click on the green square to return

□

Exercise 1(c) In this case the *least common denominator* of the fractions is 36.

$$\frac{5}{6} = \frac{5 \times 6}{6 \times 6} = \frac{30}{36},$$

$$\frac{8}{9} = \frac{8 \times 4}{9 \times 4} = \frac{32}{36},$$

$$\frac{11}{12} = \frac{11 \times 3}{12 \times 3} = \frac{33}{36},$$

The fractions are thus increasing in the order in which they appear.

Click on the green square to return   □

Exercise 1(d) In this case the *least common denominator* of the fractions is 30.

$$\frac{2}{3} \;=\; \frac{2 \times 10}{3 \times 10} \;=\; \frac{20}{30},$$

$$\frac{3}{5} \;=\; \frac{3 \times 6}{5 \times 6} \;=\; \frac{18}{30},$$

$$\frac{7}{10} \;=\; \frac{7 \times 3}{10 \times 3} \;=\; \frac{21}{30}.$$

The fractions, in increasing order of size, are $3/5$, $2/3$, $7/10$.

Click on the green square to return

□

Exercise 2(a) The *lowest common denominator* is 24, so

$$\frac{1}{3} + \frac{1}{8} = \frac{1 \times 8}{3 \times 8} + \frac{1 \times 3}{8 \times 3}$$

$$= \frac{8}{24} + \frac{3}{24}$$

$$= \frac{8+3}{24} = \frac{11}{24}.$$

Click on the green square to return   □

Exercise 2(b) Before proceeding, note that the second fraction is not in its lowest terms. Since $2/4=(1\times2)/(2\times2)=1/2$,

$$
\begin{aligned}
\frac{5}{6}+\frac{2}{4} &= \frac{5}{6}+\frac{1}{2} \\
&= \frac{5}{6}+\frac{1\times3}{2\times3} \\
&= \frac{5}{6}+\frac{3}{6} \\
&= \frac{5+3}{6}=\frac{8}{6} \\
&= \frac{4\times2}{3\times2}=\frac{4}{3}.
\end{aligned}
$$

This fraction is called an *improper fraction* since the numerator is larger than the denominator. It is perfectly acceptable as a fraction but it may also be written as $1\frac{1}{3}$.

Click on the green square to return   □

Exercise 2(c)
The *least common denominator* of the two fractions is 20 so

$$\frac{3}{4} + \frac{4}{5} = \frac{3 \times 5}{4 \times 5} + \frac{4 \times 4}{5 \times 4}$$

$$= \frac{15}{20} + \frac{16}{20}$$

$$= \frac{15 + 16}{20} = \frac{31}{20}.$$

This is another *improper fraction* which may be written as $1\frac{11}{20}$.

Click on the green square to return   □

Exercise 2(d)

The *least common denominator* of the two fractions is 12 so

$$\frac{4}{3} - \frac{1}{4} = \frac{4 \times 4}{3 \times 4} - \frac{1 \times 3}{4 \times 3}$$

$$= \frac{16}{12} - \frac{3}{12}$$

$$= \frac{16 - 3}{12} = \frac{13}{12}.$$

This is another *improper fraction* which is equal to $1\frac{1}{12}$.

Click on the green square to return ☐

Exercise 2(e)

The *least common denominator* of the two fractions is 6 so

$$\begin{aligned}
\frac{5}{6} - \frac{3}{2} &= \frac{5}{6} - \frac{3 \times 3}{2 \times 3} \\
&= \frac{5}{6} - \frac{9}{6} \\
&= \frac{5 - 9}{6} = -\frac{4}{6} \\
&= -\frac{2 \times 2}{3 \times 2} = -\frac{2}{3},
\end{aligned}$$

where the common factor 2 has been cancelled to obtain the final answer.

Click on the green square to return □

Exercise 2(f)

The *least common denominator* of the two fractions is 30 so

$$\frac{5}{6} - \frac{3}{10} = \frac{5 \times 5}{6 \times 5} - \frac{3 \times 3}{10 \times 3}$$

$$= \frac{25}{30} - \frac{9}{30}$$

$$= \frac{25 - 9}{30} = \frac{16}{30}$$

$$= \frac{8 \times 2}{15 \times 2} = \frac{8}{15},$$

where the final answer is obtained after cancellation of the common factor 2.

Click on the green square to return   □

Exercise 3(a)

$$\frac{1}{5} \times \frac{10}{11} = \frac{1}{1 \times 5} \times \frac{2 \times 5}{11},$$
$$= \frac{2}{11},$$

after cancelling the common factor 5.

Click on the green square to return □

Exercise 3(b)

Noting that division by a/b is the same as multiplication by b/a,

$$
\begin{aligned}
\frac{1}{5} \div \frac{2}{15} &= \frac{1}{5} \times \frac{15}{2}, \\
&= \frac{1}{1 \times 5} \times \frac{3 \times 5}{2}, \\
&= \frac{3}{2},
\end{aligned}
$$

after cancelling the common factor 5.

Click on the green square to return ☐

Exercise 3(c) Using exercise 2 part **(a)** we have

$$\left(\frac{1}{3} + \frac{1}{8}\right) \times \frac{2}{11} = \frac{11}{24} \times \frac{2}{11}$$
$$= \frac{2}{24} = \frac{1}{12},$$

where the common factors of 11 and then 2 have been cancelled.

Click on the green square to return   □

Exercise 3(d)

Using the result of exercise 2 part (b) we have

$$
\left(\frac{5}{6} + \frac{1}{2} \right) \div \frac{1}{3} = \frac{4}{3} \div \frac{1}{3}
$$

$$
= \frac{4}{3} \times \frac{3}{1}
$$

$$
= \frac{4}{1} = 4 \,,
$$

after recalling that division by 1/3 is the same as multiplication by 3/1=3, and cancelling the common factor 3.

Click on the green square to return   □

Exercise 3(e)

From exercise 2 part (c), the numerator is 31/20. The denominator is evaluated below.

$$\frac{1}{4} - \frac{1}{5} = \frac{1 \times 5}{4 \times 5} - \frac{1 \times 4}{5 \times 4}$$

$$= \frac{5}{20} - \frac{4}{20} = \frac{1}{20}.$$

Thus

$$\frac{\left(\frac{3}{4} + \frac{4}{5}\right)}{\left(\frac{1}{4} - \frac{1}{5}\right)} = \frac{\left(\frac{31}{20}\right)}{\left(\frac{1}{20}\right)}$$

$$= \left(\frac{31}{20}\right) \times \left(\frac{20}{1}\right)$$

$$= 31,$$

after cancellation of the common factor 20.

Click on the green square to return   □

Exercise 3(f) In this case, from exercise 2 part (c), the denominator is 13/12. Evaluating the numerator,

$$\frac{5}{6} - \frac{3}{2} = \frac{5}{6} - \frac{9}{6}$$

$$= -\frac{4}{6} = -\frac{2}{3}.$$

after cancelling the common factor 2. Thus

$$\frac{\left(\frac{5}{6} - \frac{3}{2}\right)}{\left(\frac{4}{3} - \frac{1}{4}\right)} = -\frac{\left(\frac{2}{3}\right)}{\left(\frac{13}{12}\right)} = -\left(\frac{2}{3}\right) \times \left(\frac{12}{13}\right)$$

$$= -\left(\frac{2}{3}\right) \times \left(\frac{4 \times 3}{13}\right)$$

$$= -\left(\frac{2}{1}\right) \times \left(\frac{4 \times 1}{13}\right) = -\frac{8}{13},$$

after cancellation of the common factor 3.

Click on the green square to return   □

Solutions to Quizzes

Solution to Quiz: Putting each of them in the form of a fraction with denominator 24,

$$\frac{3}{8} \;=\; \frac{3 \times 3}{8 \times 3} \;=\; \frac{9}{24},$$

$$\frac{2}{3} \;=\; \frac{2 \times 8}{3 \times 8} \;=\; \frac{16}{24},$$

$$\frac{3}{4} \;=\; \frac{3 \times 6}{4 \times 6} \;=\; \frac{18}{24}.$$

These are obviously in ascending order. Repeating this with the others will show that this is the only set in ascending order.

Solution to Quiz:
The *least common denominator* of the three fractions is 12, so

$$
\begin{aligned}
\frac{3}{4} + \frac{1}{6} &= \frac{3 \times 3}{4 \times 3} + \frac{1 \times 2}{6 \times 2} \\
&= \frac{9}{12} + \frac{2}{12} \\
&= \frac{9 + 2}{12} \\
&= \frac{11}{12}.
\end{aligned}
$$

End Quiz

Solution to Quiz: The *least common denominator* of the two fractions is 15. Writing both fractions with this denominator gives

$$\frac{2}{3} \;=\; \frac{2 \times 5}{3 \times 5} \;=\; \frac{10}{15},$$

$$\frac{4}{5} \;=\; \frac{4 \times 3}{5 \times 3} \;=\; \frac{12}{15}.$$

The fraction that is *exactly* mid-way between 10/15 and 12/15 is 11/15.   End Quiz